

ON THE FIXED POINTS OF A HAMILTONIAN DIFFEOMORPHISM IN PRESENCE OF FUNDAMENTAL GROUP

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ABSTRACT. Let M be a weakly monotone symplectic manifold, and H be a time-dependent Hamiltonian; we assume that the periodic orbits of the corresponding time-dependent Hamiltonian vector field are non-degenerate. We construct a refined version of the Floer chain complex associated to these data and any regular covering of M , and derive from it new lower bounds for the number of periodic orbits.

Using these invariants we prove in particular that if $\pi_1(M)$ is finite and solvable or simple, then the number of periodic orbits is not less than the minimal number of generators of $\pi_1(M)$. For a general closed symplectic manifold with infinite fundamental group, we show the existence of 1-periodic orbit of Conley-Zehnder index $1 - n$ for any non-degenerate 1-periodic Hamiltonian system.

1. INTRODUCTION

Let M^{2n} be a closed symplectic manifold, denote by ω its symplectic form. Let $H : \mathbb{R}/\mathbb{Z} \times M \rightarrow \mathbb{R}$ be a C^∞ function (*the Hamiltonian*). We will write $H_t(x)$ instead of $H(t, x)$. One associates to H a time-dependent Hamiltonian vector field X_{H_t} on M by the formula

$$\omega(X_{H_t}, \cdot) = dH_t \text{ for every } t.$$

Assume that every periodic orbit of $\{X_{H_t}\}$ is non-degenerate. Then the set $\mathcal{P}(H)$ of all periodic orbits is finite. Let $p(H)$ be the cardinality of this set. Denote by $\mathcal{M}(M)$ the *Morse number* of M , that is, the minimal possible number of critical points of a Morse function on M .

The celebrated Arnold conjecture (see [1], Appendix 9, and [2], p.284) says that

$$(1) \quad p(H) \geq \mathcal{M}(M).$$

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It was proved by V. I. Arnold himself in the case when H is “sufficiently small” function. The Arnold conjecture implies in particular certain homological lower bounds for $p(H)$. Namely, let us denote by $b_i(M)$ the rank of $H_i(M)$, and by $q_i(M)$ the torsion number of $H_i(M)$ (that is, the minimal possible number of generators of the abelian group $H_i(M)$). Then the conjecture (1) implies the following:

$$(2) \quad p(H) \geq \sum_i \left(b_i(M) + q_i(M) + q_{i-1}(M) \right).$$

The inequality (1) implies also the following:

$$(3) \quad p(H) \geq \sum_i b_i^{\mathbb{F}}(M),$$

where \mathbb{F} is any field and we denote by $b_i^{\mathbb{F}}(M)$ the dimension of $H_i(M, \mathbb{F})$ over \mathbb{F} .

A. Floer [8] constructed a chain complex associated with a non-degenerate 1-periodic Hamiltonian $\{H_t\}$; applying this construction he proved the homological version (3) of the Arnold Conjecture for any field \mathbb{F} in the case of monotone symplectic manifolds. (Since the degree in Floer homology is $\mathbb{Z}/2N\mathbb{Z}$, torsions in ordinary homology appearing in different degrees but congruent modulo $2N$ with relatively prime orders contribute to the Floer homology in the same degree. This is the reason why (2) does not follow from Floer homology with integer coefficients. Here N is the minimal Chern number of (M, ω) .)

The construction of the Floer chain complex was generalized to wider classes of symplectic manifolds, i.e., weakly monotone symplectic manifolds, in [12], [17]. Taking the results on orientation [8] section 2e, [9] section 21 into account, the conjecture (3) is verified in the case of weakly monotone symplectic manifolds ([8] for monotone case, [12] for $N = 0$ or $N \geq n$, [17] for weakly monotone case). If the minimal Chern number is zero, i.e., spherically Calabi-Yau, the inequality (2) holds. The construction over \mathbb{Q} was further generalized to all closed symplectic manifolds in [9], [14], hence the inequality (3) with \mathbb{F} of characteristic 0, e.g., $\mathbb{F} = \mathbb{Q}$ follows.

These results confirm the homological versions of the Arnold Conjecture, i.e., (2) holds when the minimal Chern number of the closed symplectic manifold is zero, (3) with any field \mathbb{F} holds in the case of weakly monotone closed symplectic manifolds and (3) with a field of characteristic zero holds for general closed symplectic manifolds. As for the initial conjecture (1) it is still unproved in general case. For a simply connected manifold M^{2n} with $n \geq 3$ the statement (1)

is equivalent to (2) in view of S. Smale's theorem [23]. However in the non-simply-connected case the number $\mathcal{M}(M)$ can be strictly greater than the right hand side of (2). A first step to the proof of the geometric Arnold Conjecture (1) would be to prove a weaker inequality involving only the invariants of the fundamental group. For a group G let

$$(4) \quad \mathcal{R} = \{1 \leftarrow G \leftarrow F_1 \leftarrow F_2\}$$

be a presentation of G , where F_1 and F_2 are free groups of ranks $d(\mathcal{R})$ and $r(\mathcal{R})$. Denote by $d(G)$ the minimum of numbers $d(\mathcal{R})$ for all presentations \mathcal{R} , and by $D(G)$ the minimum of numbers $d(\mathcal{R}) + r(\mathcal{R})$ for all presentations \mathcal{R} . On the occasion of Arnold-fest in Toronto 1997, V. I. Arnold asked the first author whether the development in Floer theory at that time[†] settled the original form of his conjecture, i.e., (1), and, in particular, whether one can show the following weaker assertion, which does not follow from homological version of the conjecture:

$$(5) \quad p(H) \geq D(\pi_1(M)).$$

A weaker form of this conjecture is the following:

$$(6) \quad p(H) \geq d(\pi_1(M)).$$

Since then some progress has been made in this direction, although the conjecture is far from being solved. M. Damian [5] considers similar questions in the framework of the Hamiltonian isotopies of the cotangent bundle of a compact manifold M . In a recent preprint [3] J.-F. Barraud suggested a construction of a Floer fundamental group, and proved in particular, that $p_{1-n}(H) \geq 1$ if $\pi_1(M)$ is non-trivial and M is spherical Calabi-Yau or monotone. (Here $p_j(H)$ stands for the number of periodic orbits of Conley-Zehnder index j .)

In the present paper we use the Floer chain complex associated with H and a regular covering $\tilde{M} \rightarrow M$ of the underlying manifold, and deduce from it new lower bounds for $p(H)$ in terms of certain invariants $\mu_i(\tilde{M})$, which depend on the homotopy type of M , the minimal Chern number N of M , and the chosen covering (see Definition 5.1). These invariants are similar to V. V. Sharko's invariants of chain complexes [22]. The numbers μ_i are indexed by \mathbb{N} in the case of spherical Calabi-Yau manifolds and by $\mathbb{Z}/2N\mathbb{Z}$ in case when

[†][9], [14] appeared as preprints in the previous year.

the minimal Chern number of M equals N . We have

$$p(H) \geq \sum_i \mu_i(\tilde{M}).$$

Using these invariants we obtain partial results in the direction of the conjecture (6). For a group G denote by $\delta(G)$ the minimal number of generators of the augmentation ideal of G as a $\mathbb{Z}[G]$ -module. In the case when M is weakly monotone and $\pi_1(M)$ is a finite group we prove that

$$p(H) \geq \delta(\pi_1(M)).$$

In particular we confirm the conjecture (6) for weakly monotone manifolds whose fundamental groups are finite simple or solvable (Theorem 5.7).

We also show the existence of 1-periodic orbits of Conley-Zehnder index $1 - n$ for any non-degenerate 1-periodic Hamiltonian system on any closed symplectic manifold with infinite fundamental group (Theorem 5.9).

2. FLOER COMPLEX ON THE COVERING SPACE

2.1. Novikov rings: definitions. In this preliminary subsection we gathered the definitions of several versions of the Novikov rings with which we will be working in the paper.

Let T be a finitely generated free abelian group, and $\xi : T \rightarrow \mathbb{R}$ be a homomorphism. Let R be a ring (commutative or not). Recall that the group ring $R[T]$ is the set of all finite linear combinations

$$l = \sum_{i=0}^N a_i g_i, \quad \text{with } a_i \in R, \quad g_i \in T$$

with a natural ring structure (determined by the requirement that the elements of R commute with the elements of T).

We denote by $R((T))$ the set of all formal linear combinations (infinite in general)

$$\lambda = \sum_{i=0}^{\infty} a_i g_i, \quad \text{with } a_i \in R, \quad g_i \in T$$

such that $\xi(g_i) \rightarrow -\infty$ with $i \rightarrow \infty$. Thus the series λ can be infinite, but for every C the number of terms of λ with $\xi(g_i) \geq C$ is finite. Usually the homomorphism ξ is clear from the context, so we omit it from the notation. The usual definition of the product of power series endows the abelian group $R((T))$ with the natural ring structure (we require that the elements of R commute with the elements of T).

This ring is called *the Novikov completion of the ring $R[T]$* . In this paper we will work with the case $R = \mathbb{Z}[G]$, where G is a group.

The augmentation homomorphism $\varepsilon : \mathbb{Z}[G] \rightarrow \mathbb{Z}$ has a natural extension to a ring homomorphism $R((T)) \rightarrow \mathbb{Z}((T))$, which will be denoted by the same symbol ε :

$$\varepsilon\left(\sum_i a_i g_i\right) = \sum \varepsilon(a_i) g_i \quad \text{with } a_i \in R, \quad g_i \in T.$$

Thus the ring $\mathbb{Z}((T))$ acquires a natural structure of $R((T))$ -module.

Remark 2.1. If the group G is finite, then the ring $\mathbb{Z}[G]((T))$ coincides with the group ring $\mathbb{Z}((T))[G]$ of the group G with coefficients in $\mathbb{Z}((T))$.

In the case when $\xi : T \rightarrow \mathbb{R}$ is a monomorphism, we will use abbreviated notation. The group ring $\mathbb{Z}[T]$ will be denoted by Λ , and its Novikov completion with respect to a monomorphism ξ will be denoted by $\hat{\Lambda}$. For a field \mathbb{F} we denote by \mathcal{F} the Novikov completion of the group ring $\mathbb{F}[T]$ with respect to ξ . The ring $\hat{\Lambda}$ is a principal ideal domain (PID), and \mathcal{F} is a field. We will denote the ring $\mathbb{Z}[G]((T))$ by \mathcal{L} . The ring $\mathbb{F}[G]((T))$ will be denoted by $\mathcal{L}_{\mathbb{F}}$. These rings will appear frequently in Sections 3 and 4.

The Novikov rings appear in Hamiltonian dynamics in the following context (see Subsection 2.2). Let M be a closed symplectic manifold. The de Rham cohomology class of the symplectic form determines a homomorphism $[\omega] : \pi_2(M) \rightarrow \mathbb{R}$. Consider the group

$$\Gamma = \pi_2(M)/(\ker[\omega] \cap \ker c_1(M)),$$

where $c_1(M)$ is the Chern class of the almost complex structure associated to ω . The Novikov completion $\mathbb{Z}((\Gamma))$ will be denoted by $\Lambda_{(M,\omega)}^{\mathbb{Z}}$. The Novikov ring $\mathbb{Z}[G]((\Gamma))$ will be denoted in this context by $\Lambda_{(M,\omega)}^{\mathbb{Z}[G]}$.

The restriction of the homomorphism ω to a smaller group

$$\Gamma_0 = \ker c_1(M)/(\ker[\omega] \cap \ker c_1(M))$$

is a monomorphism, so the corresponding Novikov completion $\mathbb{Z}((\Gamma_0))$ is a PID; it will be denoted by $\Lambda_{(M,\omega)}^{(0)\mathbb{Z}} = \mathbb{Z}((\Gamma_0))$. The Novikov ring $\mathbb{Z}[G]((\Gamma_0))$ will be denoted in this context by $\Lambda_{(M,\omega)}^{(0)\mathbb{Z}[G]}$.

2.2. Review on Hamiltonian Floer complex.

In this subsection, we recall the construction of Hamiltonian Floer complex with integer coefficients[†] following [8], [12], [17]. Here we use homological version. Let (M, ω) be a closed symplectic manifold of dimension $2n$. The minimal Chern number $N = N(M, \omega)$ of (M, ω) is a non-negative integer such that $\{\langle c_1(M), A \rangle | A \in \pi_2(M)\} = N\mathbb{Z}$. We call (M, ω) weakly monotone (semi-positive) if $\langle [\omega], A \rangle \leq 0$ holds for any $A \in \pi_2(M)$ with $3 - n \leq \langle c_1(M), A \rangle < 0$. This class of symplectic manifolds, in particular, contains the following.

- 1) (monotone case) We call (M, ω) a monotone symplectic manifold, if there exists a positive real number λ such that the following equality holds

$$\langle c_1(M), A \rangle = \lambda \langle [\omega], A \rangle$$

for any $A \in \pi_2(M)$.

- 2) (spherically Calabi-Yau case) We call (M, ω) spherically Calabi-Yau, if the minimal Chern number N is zero.

Let $H : \mathbb{R}/\mathbb{Z} \times M \rightarrow \mathbb{R}$ be a smooth function. Set $H_t(p) = H(t, p)$. We denote by X_{H_t} the Hamiltonian vector field of H_t . We call $\ell : \mathbb{R}/\mathbb{Z} \rightarrow M$ a periodic solution of $\{X_{H_t}\}$, if ℓ satisfies

$$\frac{d}{dt}\ell(t) = X_{H_t}(\ell(t)).$$

We assume that all contractible 1-periodic solutions of $\{X_{H_t}\}$ are non-degenerate. Denote by $\mathcal{P}(H)$ the set of contractible 1-periodic solutions of $\{X_{H_t}\}$.

Pick a generic t -dependent almost complex structure J compatible with ω . Floer chain complex $(CF_*(H, J), \delta)$ is constructed for monotone symplectic manifolds in [8] and for weakly monotone case in [12], [17].

Let $\mathcal{L}(M)$ be the space of contractible loops in M . Consider the set of pairs (ℓ, w) , where $\ell : \mathbb{R}/\mathbb{Z} \rightarrow M$ is a loop and $w : D^2 \rightarrow M$ is a bounding disk of the loop ℓ . We set an equivalence relation \sim by $(\ell, w) \sim (\ell', w')$ if and only if $\ell = \ell'$ and

$$\langle [\omega], w\#(-w') \rangle = 0 \quad \text{and} \quad \langle c_1(M), w\#(-w') \rangle = 0,$$

where $w\#(-w')$ is a spherical 2-cycle obtained by gluing w and w' with orientation reversed along the boundaries.

[†]There is an approach to construct Hamiltonian Floer complex with integer coefficients for non-degenerate periodic Hamiltonian systems on arbitrary closed symplectic manifold [10]. Since the details has not been carried out, we restrict ourselves to the class of weakly monotone symplectic manifolds.

Then the space $\overline{\mathcal{L}}(M)$ of equivalence classes $[\ell, w]$ is a covering space of $\mathcal{L}(M)$. Denote by $\Pi : \overline{\mathcal{L}}(M) \rightarrow \mathcal{L}(M)$ the covering projection, and by Γ the group of the deck transformations of this covering, so that we have

$$\Gamma = \pi_2(M)/(\ker[\omega] \cap \ker c_1(M)).$$

We have the weight homomorphism

$$\int \omega : \pi_2(M) \rightarrow \mathbb{R},$$

and the corresponding Novikov ring $\Lambda_{(M, \omega)}^{\mathbb{Z}} = \mathbb{Z}((\Gamma))$.

We define the action functional $\mathcal{A}_H : \overline{\mathcal{L}}(M) \rightarrow \mathbb{R}$ by

$$\mathcal{A}_H([\ell, w]) = \int_{D^2} w^* \omega + \int_0^1 H(t, \ell(t)) dt.$$

Then the critical point set $\text{Crit} \mathcal{A}_H$ is equal to $\Pi^{-1}(\mathcal{P}(H))$. For each pair (ℓ, w) of $\ell \in \mathcal{P}(H)$ and its bounding disk w , we have the Conley-Zehnder index $\mu_{CZ}(\ell, w) \in \mathbb{Z}$.

$$\mu_{CZ} : \text{Crit} \mathcal{A}_H \rightarrow \mathbb{Z}.$$

We define $CF_k(H, J)$ by the downward completion of the free module generated by $[\ell, w] \in \text{Crit} \mathcal{A}_H$ with $\mu_{CZ}([\ell, w]) = k \in \mathbb{Z}$ in the spirit of Novikov complex using the filtration by \mathcal{A}_H . Pick and fix a lift $[\ell, w_\ell]$ for each $\ell \in \mathcal{P}(H)$. Then $CF_*(H, J)$ is a free module generated by $[\ell, w_\ell]$ over the Novikov ring $\Lambda_{(M, \omega)}^{\mathbb{Z}}$.

The boundary operator $\partial : CF_k(H, J) \rightarrow CF_{k-1}(H, J)$ is defined by counting Floer connecting orbits.

Let $[\ell^\pm, w^\pm] \in \text{Crit} \mathcal{A}_H$. We denote by $\widetilde{\mathcal{M}}([\ell^-, w^-], [\ell^+, w^+])$ the space of the solutions $u : \mathbb{R} \times \mathbb{R}/\mathbb{Z} \rightarrow M$ satisfying

$$(7) \quad \frac{\partial u}{\partial \tau} + J(u(\tau, t)) \left(\frac{\partial u}{\partial t} - X_{H_t}(u(\tau, t)) \right) = 0$$

$$(8) \quad \lim_{\tau \rightarrow \pm\infty} u(\tau, t) = \ell^\pm(t)$$

and

$$(9) \quad [\ell^+, w^+] = [\ell^+, w^- \# u].$$

The group \mathbb{R} acts on $\widetilde{\mathcal{M}}([\ell^-, w^-], [\ell^+, w^+])$ by shifting the parametrization in τ -coordinate. We denote by $\mathcal{M}([\ell^-, w^-], [\ell^+, w^+])$ the quotient

space of $\widetilde{\mathcal{M}}([\ell^-, w^-], [\ell^+, w^+])$ by the \mathbb{R} -action. Note that the \mathbb{R} -action is free unless $[\ell^+, w^+] = [\ell^-, w^-]$. We have

$$\dim \mathcal{M}([\ell^-, w^-], [\ell^+, w^+]) = \mu_{CZ}([\ell^+, w^+]) - \mu_{CZ}([\ell^-, w^-]) - 1.$$

The moduli spaces $\mathcal{M}([\ell^-, w^-], [\ell^+, w^+])$ are oriented in a compatible way, see [8] section 2e, [9] section 21 and [18] section 5. For $[\ell^\pm, w^\pm]$ such that $\mu_{CZ}([\ell^+, w^+]) - \mu_{CZ}([\ell^-, w^-]) = 1$, $\mathcal{M}([\ell^-, w^-], [\ell^+, w^+])$ is a 0-dimensional compact oriented manifold. We denote by $n([\ell^-, w^-], [\ell^+, w^+])$ the order of $\mathcal{M}([\ell^-, w^-], [\ell^+, w^+])$ counted with signs.

For $[\ell^+, w^+] \in \text{Crit} \mathcal{A}_H$, we define

$$\partial[\ell^+, w^+] = \sum n([\ell^-, w^-], [\ell^+, w^+])[\ell^-, w^-],$$

where the summation is taken over $[\ell^-, w^-]$ such that $\mu_{CZ}([\ell^-, w^-]) = \mu_{CZ}([\ell^+, w^+]) - 1$.

In [12], [17], the Floer complex is constructed over the Novikov ring $\Lambda_{(M, \omega)}^{\mathbb{Z}/2\mathbb{Z}} \cong \Lambda_{(M, \omega)}^{\mathbb{Z}} \otimes \mathbb{Z}/2\mathbb{Z}$ with $\mathbb{Z}/2\mathbb{Z}$ -coefficients. In order to construct the Floer complex over the Novikov ring with \mathbb{Z} -coefficients, we need an appropriate coherent system of orientations on the moduli spaces $\mathcal{M}([\ell^-, w^-], [\ell^+, w^+])$. Taking [8] section 2e, [9] section 21 into account, the argument in [12] section 5 derives the well-definedness of the boundary operator ∂ and the fact that $\partial \circ \partial = 0$.

Namely, the moduli space $\mathcal{M}([\ell_1, w_1], [\ell_2, w_2])$ with $\mu_{CZ}([\ell_2, w_2]) - \mu_{CZ}([\ell_1, w_1]) = 2$ is a compact oriented 1-dimensional manifold such that boundary is the union of the direct product $\mathcal{M}([\ell_1, w_1], [\ell, w]) \times \mathcal{M}([\ell, w], [\ell_2, w_2])$ over $[\ell, w]$ with $\mu_{CZ}([\ell, w]) = \mu_{CZ}([\ell^+, w^+]) = 1$. This implies that the summation of $n([\ell_1, w_1], [\ell, w]) \cdot n([\ell, w], [\ell_2, w_2]) = 0$, hence the coefficient of $[\ell_1, w_1]$ in $\partial \circ \partial([\ell_2, w_2])$ vanishes.

Hence we have

Theorem 2.2. *Let (M, ω) be a closed weakly monotone symplectic manifold. For a non-degenerate 1-periodic Hamiltonian function H and a generic almost complex structure compatible with ω , $(CF_*(H, J), \partial)$ is a \mathbb{Z} -graded chain complex over $\Lambda_{(M, \omega)}^{\mathbb{Z}}$ with integer coefficients.*

We denote by $HF_*(H, J)$ the homology of $(CF_*(H, J), \partial)$.

Let H_α, H_β be non-degenerate 1-periodic Hamiltonians and J_α, J_β generic almost complex structures compatible with ω . Pick a one-parameter family of smooth functions $\mathcal{H} = \{H^\tau\}$ on $\mathbb{R}/\mathbb{Z} \times M$ and a one-parameter family $\mathcal{J} = \{J^\tau\}$ of almost complex structures compatible with ω such that $H^\tau = H_\alpha$ and $J^\tau = J_\alpha$ for sufficiently negative τ and $H^\tau = H_\beta$ and $J^\tau = J_\beta$ for sufficiently positive τ .

Theorem 2.3. *Let H_α, H_β and J_α, J_β be as above. Then there exists a chain homotopy equivalence*

$$\Phi_{\mathcal{H}, \mathcal{J}} : CF_*(H_\alpha, J_\alpha) \rightarrow CF_*(H_\beta, J_\beta).$$

The chain homomorphism $\Phi_{\mathcal{H}, \mathcal{J}}$ is constructed by counting *isolated* solutions $u : \mathbb{R} \times \mathbb{R}/\mathbb{Z} \rightarrow M$ joining $[\ell^-, w^-] \in \text{Crit} \mathcal{A}_{H_\alpha}$ and $[\ell^+, w^+] \in \text{Crit} \mathcal{A}_{H_\beta}$ of the following equation.

$$(10) \quad \frac{\partial u}{\partial \tau} + J^\tau(u(\tau, t)) \left(\frac{\partial u}{\partial t} - X_{H_t^\tau}(u(\tau, t)) \right) = 0.$$

For two choices $(\mathcal{H}_1, \mathcal{J}_1)$ and $(\mathcal{H}_2, \mathcal{J}_2)$, the chain homomorphisms $\Phi_{\mathcal{H}_1, \mathcal{J}_1}$ and $\Phi_{\mathcal{H}_2, \mathcal{J}_2}$ are chain homotopic. To construct a chain homotopy between them, we pick homotopies $\{\mathcal{H}_s\}$, resp. $\{\mathcal{J}_s\}$, $s \in [0, 1]$ between \mathcal{H}_1 and \mathcal{H}_2 , resp. \mathcal{J}_1 and \mathcal{J}_2 and count *isolated* solutions of Equation (10) with $\mathcal{J}_s, \mathcal{H}_s$ for some $s \in [0, 1]$.

Theorem 2.4. ([19]) *Let (M, ω) be a closed symplectic manifold and f a Morse function f on M . For a non-degenerate 1-periodic Hamiltonian H and a generic almost complex structure J compatible with ω , the Floer complex $(CF_*(H, J), \partial)$ is chain equivalent to the Morse complex $(CM_{*+n}(f) \otimes \Lambda_{(M, \omega)}^\mathbb{Z}, \partial^{\text{Morse}})$.*

For the comparison of orientation of the moduli space of solutions of Equation (7) and the moduli space of Morse gradient flow lines, see [9] section 21.

Remark 2.5. If (M, ω) is either monotone, spherically Calabi-Yau or the minimal Chern number $N \geq n$, we have a chain homotopy equivalence between $(CF_*(H, J), \partial)$ and $(CF_*(f, J), \partial)$ for a sufficiently small Morse function f . The latter is isomorphic to the Morse complex $(CM_{*+n}(f) \otimes \Lambda_{(M, \omega)}^\mathbb{Z}, \partial^{\text{Morse}})$, see [12] Proposition 7.4.

In [17], we introduced modified Floer homology $\widehat{HF}_*(H, J)$, which is computed in the case that (M, ω) is a closed weakly monotone symplectic manifold and show that

$$\widehat{HF}_*(H, J) \cong H_{*+n}(M; \Lambda_{(M, \omega)}^\mathbb{Z}).$$

(In order to work over integer coefficients, we use the orientation of the moduli space of solutions of Equation (7), (10) as in [9] section 21.) In the end of section 6.3 [11], we have an isomorphism between $\widehat{HF}_*(H, J)$ and $HF_*(H, J)$, see also [4] Remark 4, which also yields

$$HF_*(H, J) \cong H_{*+n}(M; \Lambda_{(M, \omega)}^\mathbb{Z}).$$

If the minimal Chern number of (M, ω) is N , the Floer chain complex $(CF_*(H, J), \delta)$ is $2N$ -periodic, i.e.,

$$(CF_*, \partial) \cong (CF_{*+2N}, \partial).$$

(Pick an element $A \in \pi_2(M)$ such that $\langle c_1(M), A \rangle = N$. Then the action of $[A] \in \pi_2(M)/(\ker[\omega] \cap \ker c_1(M))$ induces such an isomorphism of chain complexes.) The $\mathbb{Z}/2N\mathbb{Z}$ -graded version of Floer complex (CF_*, δ) is a free finitely generated chain complex over the smaller Novikov ring. Namely, put

$$\Gamma_0 = \ker c_1(M)/(\ker[\omega] \cap \ker c_1(M)).$$

endow it with the homomorphism $\int \omega : \Gamma_0 \rightarrow \mathbb{R}$ and consider the corresponding Novikov completion $\Lambda_{(M, \omega)}^{(0)\mathbb{Z}} = \mathbb{Z}((\Gamma_0))$. We may also denote this ring by $\hat{\Lambda}$ (observe that $\int \omega : \Gamma_0 \rightarrow \mathbb{R}$ is a monomorphism). We have the following

Theorem 2.6. *Let (M, ω) be a closed weakly monotone symplectic manifold with minimal Chern number N and H a non-degenerate 1-periodic Hamiltonian on M .*

- (1) *Then there exists a $\mathbb{Z}/2N\mathbb{Z}$ -graded chain complex $(CF_*^\circ(H, J), \partial)$, which is freely generated by $\{[\ell, w_\ell] \mid \ell \in \mathcal{P}(H)\}$ over $\Lambda_{(M, \omega)}^{(0)\mathbb{Z}}$.*
- (2) *(CF_*°, ∂) is chain equivalent to the Morse complex $(CM_{*+n}^\circ, \partial^{\text{Morse}})$ with the grading modulo $2N$ as chain complexes over $\Lambda_{(M, \omega)}^{(0)\mathbb{Z}}$.*

We can also construct Floer complex with coefficients in a local system on M , see [18] section 6, and prove corresponding results in the case with coefficients in a local system on M .

Theorem 2.7. *Let (M, ω) be a closed weakly monotone symplectic manifold and $\rho : \pi_1(M) \rightarrow GL(r, \mathbb{F})$. For a non-degenerate 1-periodic Hamiltonian H and a generic almost complex structure compatible with ω ,*

$$HF_*^\circ(H, J; \rho) \cong H_{*+n}^\circ(M; \rho) \otimes \Lambda_{(M, \omega)}^\mathbb{F}.$$

2.3. Floer complex over a regular cover. Let $pr : \tilde{M} \rightarrow M$ be a regular covering of M with the covering transformation group G . Let H be a non-degenerate 1-periodic Hamiltonian M and J a generic t -dependent almost complex structures compatible with ω . Denote by \tilde{H} , resp. \tilde{J} the pull-back of H , resp. J , to $\mathbb{R}/\mathbb{Z} \times \tilde{M}$. The Floer complex $(CF_*(\tilde{H}, \tilde{J}), \partial)$ is constructed in the spirit of [13].

We will now define $\tilde{\mathcal{L}}(\tilde{M})$ similarly to $\tilde{\mathcal{L}}(M)$. Consider the set of all pairs (γ, w) where γ is a loop in \tilde{M} and w is a bounding disk for

$pr \circ \gamma$. Introduce in this set the following equivalence relation: (γ, w) and (γ', w') are equivalent if the values of both cohomology classes $[\omega]$ and $c_1(M)$ on the singular sphere $w \# (-w')$ are the same. The set $\overline{\mathcal{L}}(\widetilde{M})$ of the equivalence classes is a covering space of $\mathcal{L}(\widetilde{M})$, and the deck transformation group of the covering is isomorphic to

$$\Gamma = \pi_2(M)/(\ker[\omega] \cap \ker c_1(M)).$$

The action functional

$$\mathcal{A}_{\widetilde{H}} : \overline{\mathcal{L}}(\widetilde{M}) \rightarrow \mathbb{R}$$

is defined by the same formula as before, namely

$$\mathcal{A}_{\widetilde{H}}(\gamma, w) = \mathcal{A}_H(pr \circ \gamma, w).$$

We define $CF_*(\widetilde{H}, \widetilde{J})$ by the downward completion of free abelian group generated by $\text{Crit} \mathcal{A}_{\widetilde{H}}$ with respect to the action functional $\mathcal{A}_{\widetilde{H}}$. Pick and fix a lift $\tilde{\ell}$ of $\ell \in \mathcal{P}(H)$ to a 1-periodic solution of $X_{\widetilde{H}_t}$ on \widetilde{M} . Note that $pr^{-1}(\mathcal{P}(H)) = \{g \cdot \tilde{\ell} | \ell \in \mathcal{P}(H), g \in G\}$. We have

$$\text{Crit} \mathcal{A}_{\widetilde{H}} = \{[g \cdot \tilde{\ell}, w] | [\ell, w] \in \text{Crit} \mathcal{A}_H, g \in G\}.$$

We also pick and fix a bounding disk w_ℓ for each $\ell \in \mathcal{P}(H)$. Then we find that $CF_*(\widetilde{H}, \widetilde{J})$ is isomorphic to a free module generated by $\{[\tilde{\ell}, w_\ell] | \ell \in \mathcal{P}(H)\}$ over $\Lambda_{(M, \omega)}^{\mathbb{Z}[G]}$.

The boundary operator is defined by counting certain isolated solutions of Equation (7) as follows. Let $[\gamma^\pm, w^\pm] \in \text{Crit}_{\widetilde{H}}$. We consider the moduli space $\mathcal{M}([\gamma^-, w^-], [\gamma^+, w^+])$ of solutions of Equation (7) satisfying Condition (8) with $\ell^\pm = pr \circ \gamma^\pm$, Condition (9) and that $u(\tau, 0) : \mathbb{R} \rightarrow \widetilde{M}$ lifts to a path joining $\gamma^-(0)$ and $\gamma^+(0)$. We set $n([\gamma^-, w^-], [\gamma^+, w^+])$ the signed count of isolated solutions in $\mathcal{M}([\gamma^-, w^-], [\gamma^+, w^+])$. Since isolated connecting orbits in $\mathcal{M}([pr \circ \gamma^-, w^-], [pr \circ \gamma^+, w^+])$ are at most finitely many $\mathcal{M}([\gamma^-, w^-], [g \cdot \gamma^+, w^+])$ contains isolated connecting orbits. In other words, for fixed $[\gamma^-, w^-]$, $[\gamma^+, w^+]$, there are at most finitely many $g \in G$ such that $n([\gamma^-, w^-], [g \cdot \gamma^+, w^+]) \neq 0$. The boundary operator ∂ on $CF_*(\widetilde{H}, \widetilde{J})$ is given by

$$\partial[\gamma^+, w^+] = \sum n([\gamma^-, w^-], [\gamma^+, w^+])[\gamma^-, w^-].$$

It is clear that ∂ is linear over $\Lambda_{(M, \omega)}^{\mathbb{Z}[G]}$.

Keeping attention on the homotopy classes of paths $u(\tau, 0) : \mathbb{R} \rightarrow M$ of solutions of Equation (7), (10), the proofs of Theorems 2.2, 2.3 and 2.4 works for the case of $CF_*(\widetilde{H}, \widetilde{J})$. For example, we show the fact that $\partial \circ \partial = 0$ in the following way. In subsection 2.2, we recalled that $(CF_*(H, J), \partial)$ is a chain complex using the moduli space

$\mathcal{M}([\ell_1, w_1], [\ell_2, w_2])$ with $\mu_{CZ}([\ell_2, w_2]) - \mu_{CZ}([\ell_1, w_1]) = 2$. The projection $pr : \tilde{M} \rightarrow M$ gives an identification of $\mathcal{M}([\gamma_1, w_1], [\gamma_2, w_2])$ and the subspace of $\mathcal{M}(pr \circ \gamma^-, w^-, pr \circ \gamma^+, w^+)$ consisting of connecting orbits u such that $u(\tau, 0)$ lifts to a path joining $\gamma_1(0)$ and $\gamma_2(0)$. The boundary of this subspace is the union of the direct product $\mathcal{M}([\gamma_1, w_1], [\gamma, w]) \times \mathcal{M}([\gamma, w], [\gamma_2, w_2])$ such that $\mu_{CZ}([\gamma, w]) = \mu_{CZ}([\gamma^+, w^+]) - 1$, which is identified with the union of the space of pairs (u_1, u_2) of $\mathcal{M}(pr \circ \gamma_1, w_1, pr \circ \gamma, w) \times \mathcal{M}([\partial \circ \gamma, w], pr \circ \gamma_2, w_2]$ such that the concatenation of the paths $u_1(\tau, 0)$ and $u_2(\tau, 0)$ lifts to a path joining $\gamma_1(0)$ and $\gamma_2(0)$. Hence, by looking at the components of $\mathcal{M}(pr \circ \gamma_1, w_1, pr \circ \gamma_2, w_2]$ with $u(\tau, 0)$ in the prescribed homotopy class of paths joining $pr \circ \gamma_1(0)$ and $pr \circ \gamma_2(0)$, we find that $CF_*(\tilde{H}, \tilde{J})$ is a chain complex. This Floer complex is periodic with respect to the degree shift by $2N$. Hence we can also obtain $\mathbb{Z}/2N\mathbb{Z}$ -graded chain complex, which we denote by $CF_*^\circ(H, J)$, $CF_*^\circ(\tilde{H}, \tilde{J})$, etc.

Theorem 2.8. *Let $\tilde{M} \rightarrow M$ be a regular covering of a closed weakly monotone symplectic manifold (M, ω) with minimal Chern number N . Let G be the structure group of the covering.*

- (1) *For a non-degenerate 1-periodic Hamiltonian H on M , there exists a $\mathbb{Z}/2N\mathbb{Z}$ -graded chain complex $(CF_*^\circ(\tilde{H}, \tilde{J}), \partial)$ such that $CF_*^\circ(\tilde{H}, \tilde{J})$ is a free module generated by $\{[\tilde{\ell}, w_\ell] | \ell \in \mathcal{P}(H)\}$ over $\Lambda_{(M, \omega)}^{(0)\mathbb{Z}[G]}$.*
- (2) *Let f be a Morse function on M . Then for a non-degenerate 1-periodic Hamiltonian H and a generic almost complex structure J compatible with ω , $(CF_*^\circ(\tilde{H}, \tilde{J}), \partial)$ is chain equivalent to the Morse complex $(CM_{*+n}^\circ(f \circ pr) \otimes_{\mathbb{Z}[G]} \Lambda_{(M, \omega)}^{(0)\mathbb{Z}[G]}, \partial^{\text{Morse}})$ of the Morse function $f \circ pr : \tilde{M} \rightarrow \mathbb{R}$ with coefficients in $\Lambda_{(M, \omega)}^{(0)\mathbb{Z}[G]}$.*

In the case of arbitrary closed symplectic manifolds we have an analog of this result over the field \mathbb{Q} .

Theorem 2.9. *Let $\tilde{M} \rightarrow M$ be a regular covering of a closed symplectic manifold (M, ω) with minimal Chern number N . Let G be the structure group of the covering.*

- (1) *For a non-degenerate 1-periodic Hamiltonian H on M , there exists a $\mathbb{Z}/2N\mathbb{Z}$ -graded chain complex $(CF_*^\circ(\tilde{H}, \tilde{J}), \partial)$ such that $CF_*^\circ(\tilde{H}, \tilde{J})$ is a free module generated by $\{[\tilde{\ell}, w_\ell] | \ell \in \mathcal{P}(H)\}$ over $\Lambda_{(M, \omega)}^{(0)\mathbb{Q}[G]}$.*
- (2) *Let f be a Morse function on M . Then for a non-degenerate 1-periodic Hamiltonian H and a generic almost complex structure J compatible with ω , $(CF_*^\circ(\tilde{H}, \tilde{J}), \partial)$ is chain equivalent to the Morse*

complex $(CM_{*+n}^\circ(f \circ pr) \otimes_{\mathbb{Q}[G]} \Lambda_{(M,\omega)}^{(0)\mathbb{Q}[G]}, \partial^{\text{Morse}})$ of the Morse function $f \circ pr : \widetilde{M} \rightarrow \mathbb{R}$ with coefficients in $\Lambda_{(M,\omega)}^{(0)\mathbb{Q}[G]}$.

3. INVARIANTS OF CHAIN COMPLEXES: \mathbb{Z} -GRADED CASE

The sections 3 and 4 are purely algebraic. We introduce some invariants of chain complexes, which will be applied in Section 5 to obtain lower bounds for $p(H)$.

3.1. Definition of invariants μ_i .

Recall that a ring R is called an *IBN-ring*, if the cardinality of a base of a free R -module does not depend on the choice of the base. Any principal ideal domain (PID) is an IBN-ring. The group ring of any group with coefficients in a PID is an IBN-ring. All the rings which we consider in this paper will be IBN-rings.

Definition 3.1. For a free based finitely generated module A over an IBN-ring R we denote by $m(A)$ the cardinality of any base of A . It will be called *the rank of A* .

Let $C_* = \{C_n\}_{n \in \mathbb{Z}}$ be a free finitely generated chain complex over a ring R . Denote by $m_i(C_*)$ the number $m(C_i)$. The minimum of the numbers $m_i(D_*)$, where D_* ranges over the set of all free based finitely generated chain complexes chain equivalent to C_* , will be denoted by $\mu_i(C_*)$.

Observe that the chain complexes which we consider are not supposed to vanish in negative degrees.

Our aim in this section is to develop efficient tools for computing the invariants $\mu_i(C_*)$ for the case of chain complexes arising in the applications to the Arnold conjecture.

We will use here the terminology from Subsection 2.1. Namely G is a group, T is a free abelian finitely generated group, $\xi : T \rightarrow \mathbb{R}$ is a monomorphism. We consider the rings

$$\mathcal{L} = \mathbb{Z}[G]((T)), \quad \hat{\Lambda} = \mathbb{Z}((T)), \quad \mathcal{F} = \mathbb{F}((T)).$$

The ring \mathcal{L} is an IBN-ring since it has an epimorphism onto $\hat{\Lambda}$. Similarly, $\mathcal{L}_{\mathbb{F}}$ is an IBN-ring.

We will use the following notation throughout the rest of the paper:

X is a connected finite CW-complex, $\widetilde{X} \rightarrow X$ is a regular covering with a structure group G , so that we have an epimorphism $\pi_1(X) \rightarrow G$,

$$(11) \quad \mathcal{C}_*(X) = C_*(\widetilde{X}) \otimes_{\mathbb{Z}G} \mathbb{Z}[G]((T)).$$

Then $\mathcal{C}_*(X)$ is a free finitely generated chain complex over $\mathcal{L} = \mathbb{Z}[G]((T))$. Observe the following isomorphism of \mathcal{L} -modules:

$$(12) \quad H_0(\mathcal{C}_*(X)) \approx \hat{\Lambda}.$$

3.2. Lower bounds provided by the cohomology with local coefficients.

Let $\rho : G \rightarrow \mathrm{GL}(r, \mathbb{F})$ be a representation. We denote by $b_i(X, \rho)$ the Betti numbers of X with respect to the local coefficient system induced by ρ . Put

$$(13) \quad \beta_i(X, \rho) = \frac{1}{r} b_i(X, \rho).$$

We will need the following basic Lemma.

Lemma 3.2. *Let D_* be a free finitely generated chain complex over \mathcal{L} , chain equivalent to $\mathcal{C}_*(X)$. Let $\rho : G \rightarrow \mathrm{GL}(r, \mathbb{F})$ be a representation. Then there is a chain complex E_* over \mathcal{F} such that*

- 1) $\dim_{\mathcal{F}} E_k = r \cdot m_k(D_*)$,
- 2) $\dim_{\mathcal{F}} H_k(E_*) = b_k(X, \rho)$.

Proof. The vector space \mathbb{F}^r has the structure of $\mathbb{Z}[G]$ -module via the representation ρ . This structure induces a natural structure $\hat{\rho}$ of \mathcal{L} -module on \mathcal{F}^r by the following formula

$$\left(\sum_i a_i g_i \right) \left(\sum_j v_j h_j \right) = \sum_{i,j} a_i (v_j) g_i h_j \quad \text{with } a_i \in \mathbb{Z}[G], \quad v_j \in \mathbb{F}^r, \quad g_i, h_j \in T.$$

We have also the representation $\rho_0 : G \rightarrow \mathrm{GL}(r, \mathcal{F})$ obtained as the composition of ρ with the embedding $\mathbb{F} \hookrightarrow \mathcal{F}$. Put $E_* = D_* \otimes_{\hat{\rho}} \mathcal{F}^r$.

Then $\dim_{\mathcal{F}} E_k = r \cdot m_k(D_*)$, and

$$\mathcal{C}_*(X) \otimes_{\hat{\rho}} \mathcal{F}^r = \left(C_*(\tilde{X}) \otimes_{\mathbb{Z}[G]} \mathcal{L} \right) \otimes_{\hat{\rho}} \mathcal{F}^r \approx C_*(\tilde{X}) \otimes_{\rho_0} \mathcal{F}^r \approx \left(C_*(\tilde{X}) \otimes_{\rho} \mathbb{F}^r \right) \otimes_{\mathbb{F}} \mathcal{F}$$

so that

$$\dim_{\mathcal{F}} H_k(E_*) = \dim_{\mathcal{F}} H_k \left(\mathcal{C}_*(X) \otimes_{\hat{\rho}} \mathcal{F}^r \right) = b_k(X, \rho). \quad \square$$

The next proposition follows.

Proposition 3.3. *We have*

$$\mu_i(\mathcal{C}_*(X)) \geq \beta_i(X, \rho).$$

Proof. Let D_* be any chain complex over \mathcal{L} which is chain equivalent to $\mathcal{C}_*(X)$. Pick a chain complex E_* constructed in the previous Lemma. We have

$$r \cdot m_k(D_*) = \dim_{\mathcal{F}} E_k \geq \dim_{\mathcal{F}} H_k(E_*) = b_k(X, \rho) = r \cdot \beta_i(X, \rho). \quad \square$$

For $i = 1$ we have a slightly stronger version of this estimate. In order to prove it, we need a lemma.

Lemma 3.4. *Let A_* be a free based finitely generated chain complex over a ring R , such that $A_* = 0$ for $* \leq l - 2$ and $H_{l-1}(A_*) = 0$. Then there exists a chain complex B_* such that*

- 1) $B_j \approx A_j$ for $j \geq l + 2$, and $j = l$.
- 2) $B_{l+1} = A_{l+1} \oplus A_{l-1}$,
- 3) $B_j = 0$ for $j \leq l - 1$.

Proof. Let T_* be the chain complex

$$\{0 \leftarrow A_{l-1} \xleftarrow{Id} A_{l-1} \leftarrow 0\}$$

concentrated in degrees l and $l + 1$. By the Thickening Lemma ([22], Lemma 3.6, p. 56) the chain complex $C_* = A_* \oplus T_*$ is isomorphic to the chain complex

$$C'_* = \{0 \leftarrow A_{l-1} \xleftarrow{\partial_l} A_{l-1} \oplus A_l \xleftarrow{\partial_{l+1}} A_{l-1} \oplus A_{l+1} \leftarrow A_{l+2} \leftarrow \dots\}$$

with $\partial_l(x, y) = x$. Splitting off the chain complex

$$\{0 \leftarrow A_{l-1} \xleftarrow{Id} A_{l-1} \leftarrow 0\}$$

concentrated in degrees $l - 1$ and l , we obtain the required chain complex B_* . \square

Proposition 3.5. *We have*

$$\mu_1(\mathcal{C}_*(X)) \geq \beta_1(X, \rho) + 1$$

for any representation ρ such that $H_0(X, \rho) = 0$.

Proof. Let D_* be any free based finitely generated chain complex over \mathcal{L} , chain equivalent to $\mathcal{C}_*(X)$. An easy induction argument using Lemma 3.4 shows that D_* is chain equivalent to a free based finitely generated chain complex D'_* such that $D'_i = 0$ for $i \leq -2$ and $D'_i = D_i$ for $i \geq 1$. Put

$$\alpha = m(D'_{-1}), \quad \beta = m(D'_0), \quad \gamma = m(D'_1).$$

The homology of D'_* and of $D_* \otimes_{\hat{\Lambda}_{\mathbb{F}}^r} \rho$ vanishes in degree -1 . Applying Lemma 3.2 to the trivial 1-dimensional representation ρ_0 we find a chain complex

$$E_* = \{0 \longleftarrow \mathcal{F}^\alpha \xleftarrow{\partial_0} \mathcal{F}^\beta \xleftarrow{\partial_1} \mathcal{F}^\gamma \longleftarrow \dots\}$$

such that $H_{-1}(E_*) = 0$ and $H_0(E_*) \approx \mathcal{F}$; this implies $\beta \geq \alpha + 1$. Applying Lemma 3.2 to the representation ρ we obtain a chain complex

$$E'_* = \{0 \longleftarrow \mathcal{F}^{\alpha r} \xleftarrow{\partial'_0} \mathcal{F}^{\beta r} \xleftarrow{\partial'_1} \mathcal{F}^{\gamma r} \longleftarrow \dots\}$$

with $\dim_{\mathcal{F}} \text{Ker } \partial'_1 \geq b_1(X, \rho)$. Since $H_0(X, \rho)$ and $H_0(E'_*)$ vanish, we obtain

$$r\gamma \geq b_1(X, \rho) + r(\beta - \alpha) \geq b_1(X, \rho) + r. \quad \square$$

Remark 3.6. Observe that the condition $H_0(X, \rho) = 0$ is true for every non-trivial irreducible representation ρ .

When $\pi_1(X)$ is a perfect group the above methods allow to obtain a lower bound for $\mu_2(X)$:

Proposition 3.7. *Assume that the covering $\widetilde{X} \rightarrow X$ is the universal covering of X , so that in particular $\pi_1(X) \approx G$. Assume that G is a perfect finite group. Then $\mu_2(X) \geq b_2(X) + 2$.*

Proof. Let D_* be any free based finitely generated chain complex over \mathcal{L} , chain equivalent to $\mathcal{C}_*(X)$. Consider the chain complex D'_* constructed in the proof of the Lemma 3.5. Applying Lemma 3.2 to the trivial 1-dimensional representation ρ_0 we find a chain complex

$$E_* = \{0 \longleftarrow \mathcal{F}^\alpha \xleftarrow{\partial_0} \mathcal{F}^\beta \xleftarrow{\partial_1} \mathcal{F}^\gamma \xleftarrow{\partial_2} \mathcal{F}^\delta \longleftarrow \dots\}$$

such that $H_{-1}(E_*) = 0$ and $H_0(E_*) \approx \mathcal{F}$; We have $\dim \text{Ker } \partial_1 = \gamma - (\beta - \alpha) + 1$. Since G is perfect, we have $\dim H_1(E_*) = b_1(X) = 0$, therefore $\dim \text{Im } \partial'_2 = \gamma - (\beta - \alpha) + 1$, so that

$$\delta \geq b_2(X) + \gamma - (\beta - \alpha) + 1.$$

Choose an irreducible representation ρ , such that $b_1(X, \rho) \geq 1$ (this is possible by Theorem 3.8). Applying Lemma 3.2 to the representation ρ we deduce $\gamma - (\beta - \alpha) \geq \beta_1(X, \rho) > 0$, so that $\delta \geq b_2(X) + 2$. \square

3.3. The invariant μ_1 : the case of finite groups. The results of the previous subsection allow to obtain a complete result for the invariant μ_1 in the case of finite groups. For a group G denote by $d(G)$ the minimal possible number of generators of G and by $\delta(G)$ the minimal possible number of generators of the augmentation ideal of $\mathbb{Z}[G]$ as a $\mathbb{Z}[G]$ -module. The next theorem is a reformulation of a well-known result in the cohomological theory of finite groups (see, for example, [21], Corollary 5.8, p. 191).

Theorem 3.8. *Let G be a finite group. Then $\delta(G)$ equals the maximum of two numbers $A(G)$ and $B(G)$, defined below.*

$$A(G) = \max_{p, \rho} \left(\left\lfloor \frac{1}{r} b_1(G, \rho) \right\rfloor + 1 \right),$$

where the maximum is taken over all prime divisors p of $|G|$ and all the irreducible non-trivial representations $\rho : G \rightarrow \mathrm{GL}(r, \mathbb{F}_p)$.[†]

$$B(G) = \max_p b_1(G, \mathbb{F}_p),$$

where the maximum is taken over all prime divisors p of $|G|$. \square

Remark 3.9. If $\phi : G \rightarrow K$ is a group epimorphism, V a K -module, and we endow V with a structure of G -module via ϕ , then the induced homomorphism $H_1(G, V) \rightarrow H_1(K, V)$ is surjective.

Recall from (11) that X denotes a connected finite CW-complex, and $\tilde{X} \rightarrow X$ is a regular covering with a structure group G , so that we have an epimorphism $\pi_1(X) \rightarrow G$. We denote by $\mathcal{C}_*(X)$ the chain complex $C_*(\tilde{X}) \otimes_{\mathbb{Z}G} \mathcal{L}$.

Theorem 3.10. *Let G be a finite group with a group epimorphism $\pi_1(X) \rightarrow G$. Then $\mu_1(\mathcal{C}_*(X)) \geq \delta(G)$.*

Proof. Let D_* be a free based finitely generated chain complex chain equivalent to $\mathcal{C}_*(X)$. The inequality $\mu_1(\mathcal{C}_*(X)) \geq \delta(G)$ follows immediately from Propositions 3.3, 3.5 together with Theorem 3.8 and Remarks 3.6, 3.9. \square

The invariant $\delta(G)$ of a finite group has the following properties:

- 1) $\delta(G) = d(G)$ if G is solvable (K. Grünberg's theorem [6], see also [21], Theorem 5.9).
- 2) $\delta(G) = 1$ if and only if G is cyclic (see [21], Lemma 5.5).

The second point implies also that $\delta(G)$ equals 2 for any simple non-abelian group G (since $d(G) = 2$ for such a group).

[†]The symbol $\lfloor z \rfloor$ denotes the minimal integer $k \geq z$.

Corollary 3.11. *Let G be a finite group with a group epimorphism $\pi_1(X) \rightarrow G$. We have*

- 1) $\mu_1(\mathcal{C}_*(X)) \geq d(G)$ if G is solvable or simple.
- 2) $\mu_1(\mathcal{C}_*(X)) \geq 1$ for every non-trivial group G .
- 3) $\mu_1(\mathcal{C}_*(X)) \geq 2$ if G is not cyclic.

3.4. The invariant μ_1 : the case of infinite groups. If the group G is infinite, it is more difficult to give computable lower bounds for $\mu_1(\mathcal{C}_*(X))$. In this section we prove that $\mu_1(\mathcal{C}_*(X)) > 0$. Recall the ring $\mathcal{L} = \mathbb{Z}[G]((T))$ and its subring $\hat{\Lambda} = \mathbb{Z}((T))$. Observe that the $\hat{\Lambda}$ -module \mathcal{L} has no torsion. The ring $\hat{\Lambda}$ is also a module over \mathcal{L} via the augmentation homomorphism ε (see Subsection 3.1).

Lemma 3.12. *A free \mathcal{L} -module contains no submodule isomorphic to $\hat{\Lambda}$.*

Proof. Assume that there is an embedding $i : \hat{\Lambda} \rightarrow \mathcal{L}^n$. There is a projection $p : \mathcal{L}^n \rightarrow \mathcal{L}$ such that $p \circ i$ is non-trivial. Since \mathcal{L} has no $\hat{\Lambda}$ -torsion, the homomorphism $p \circ i$ is an embedding. Put $a = (p \circ i)(1) \in \mathcal{L}$. Multiplying a by a suitable element of T if necessary, we can consider that $a = \alpha \cdot 1 + \alpha'$ where $\alpha \in \mathbb{Z}[G]$, and α' is a power series in monomials $g_i \in T$ with $\xi(g_i) < 0$. For every $g \in G$ we have then $g \cdot a = \lambda \cdot a$ with some $\lambda \in \hat{\Lambda}$, which implies $g\alpha = l\alpha$ for some $l \in \mathbb{Z}$. The last property is impossible, since G is infinite, and α is a finite linear combination of elements of G . \square

Corollary 3.13. *Let D_* be a free based finitely generated chain complex over \mathcal{L} . Assume that $H_0(D_*) \approx \hat{\Lambda}$. Then $D_1 \neq 0$.*

Proof. If $D_1 = 0$, then $H_0(D_*)$ is isomorphic to the kernel of the boundary operator $\partial_0 : D_0 \rightarrow D_{-1}$, therefore $H_0(D_*)$ is a submodule of free \mathcal{L} -module, which contradicts to Lemma 3.12. \square

The next proposition follows.

Proposition 3.14. *If G is infinite, then $\mu_1(\mathcal{C}_*(X)) \geq 1$.*

Remark 3.15. This proposition is valid also for non-trivial finite groups, see Theorem 3.10.

3.5. The invariant μ_1 : the general case.

Definition 3.16. Let R be a ring, and N be a module over R . The minimal number s such that there exists an epimorphism $R^{s+r} \rightarrow N \oplus R^r$ will be called the *stable number of generators of N* , and denoted by $\sigma(N)$. The stable number of generators of the augmentation ideal $\text{Ker } \varepsilon : \mathcal{L} \rightarrow \hat{\Lambda}$ of the ring \mathcal{L} will be denoted $\sigma(G)$.

Remark 3.17. Using Theorem 3.8 it is easy to show that $\sigma(G) = \delta(G)$ for any finite group. It does not seem that this equality holds in general, although we do not have a counter-example at present.

Proposition 3.18. $\mu_1(\mathcal{C}_*(X)) \geq \sigma(G)$.

Proof. Let D_* be any free based finitely generated chain complex over \mathcal{L} , chain equivalent to $\mathcal{C}_*(X)$. Similarly to Proposition 3.5 we construct a free based finitely generated chain complex D'_* chain equivalent to D_* such that $D'_i = 0$ for $i \leq -2$ and $D'_i = D_i$ for $i \geq 1$. Denote D'_{-2} by A and D'_{-1} by B . Similarly to the proof of 3.5 we deduce $m(D_0) \geq m(B) - m(A) + 1$. We have $H_i(D_*) = 0$ for $i < 0$. Applying Lemma 3.4 we obtain a chain complex

$$D'_* = \{ \dots 0 \leftarrow D'_{-2} \xleftarrow{\partial_{-1}} D'_{-1} \leftarrow D_0 \leftarrow D_1 \leftarrow \dots \}$$

Applying Lemma 3.4 two more times, we obtain a chain complex

$$D''_* = \{ \dots 0 \leftarrow D_0 \oplus A \leftarrow D_1 \oplus B \leftarrow D_2 \leftarrow \dots \}$$

chain equivalent to D'_* . Since $H_0(D'_*) \approx \hat{\Lambda}$, we have an exact sequence

$$0 \leftarrow \hat{\Lambda} \xleftarrow{\phi} D_0 \oplus A \xleftarrow{\psi} D_1 \oplus B.$$

Add to it the exact sequence $\{0 \leftarrow 0 \leftarrow \mathcal{L} \xleftarrow{Id} \mathcal{L} \leftarrow 0\}$. By the Thickening Lemma the result is isomorphic to the following exact sequence

$$0 \leftarrow \hat{\Lambda} \xleftarrow{\chi} \mathcal{L} \oplus D_0 \oplus A \xleftarrow{\phi} \mathcal{L} \oplus D_1 \oplus B$$

where $\chi(f, d, a) = \varepsilon(f)$. Let $J(G) = \text{Ker}(\varepsilon : \mathcal{L} \rightarrow \hat{\Lambda})$. We have $\text{Ker} \chi = J(G) \oplus D_0 \oplus A$, so that ϕ is an epimorphism of a free \mathcal{L} -module of rank $m(B) + m(D_1) + 1$ onto the sum of $J(G)$ and a free \mathcal{L} -module of rank $m(A) + m(D_0)$. Thus

$$\sigma(G) \leq m(B) - m(A) + 1 - m(D_0) + m(D_1) \leq m(D_1). \quad \square$$

3.6. The invariant μ_2 . The results about this invariant are less complete, than for μ_1 : we have two different lower bounds for $\mu_2(\mathcal{C}_*(X))$ (Proposition 3.19 and Corollary 3.28), none of them is optimal in general. Denote by $B_1(X)$ the maximum of numbers $\beta_1(X, \rho) - \beta_0(X, \rho)$ where ρ ranges over all representations of G .

Proposition 3.19. For every representation $\rho : G \rightarrow \text{GL}(r, \mathbb{F})$ we have

$$(14) \quad \mu_2(\mathcal{C}_*(X)) \geq B_1(X) + \beta_2(X, \rho) - \beta_1(X, \rho) + \beta_0(X, \rho).$$

Proof. Let D_* be any free finitely generated chain complex over \mathcal{L} , chain equivalent to $\mathcal{C}_*(X)$. Similarly to Proposition 3.5 we can assume that $D_i = 0$ for $i \leq -2$. Put

$$\alpha = m(D_{-1}), \beta = m(D_0), \gamma = m(D_1), \delta = m(D_2).$$

Apply Lemma 3.2 and let E_* be the corresponding chain complex. Denote by Z_0 the space of cycles of degree 0 of this complex; then $\dim_{\mathcal{F}} Z_0 = r(\beta - \alpha)$. Consider the chain complex

$$0 \leftarrow Z_0 \leftarrow E_1 \leftarrow E_2 \leftarrow \dots$$

of vector spaces over \mathcal{F} . Its Betti numbers are equal to the Betti numbers of X with coefficients in ρ , and applying the strong Morse inequalities we obtain the following:

$$(15) \quad \beta - \alpha \geq \beta_0(X, \rho);$$

$$(16) \quad \gamma - (\beta - \alpha) \geq \beta_1(X, \rho) - \beta_0(X, \rho);$$

$$(17) \quad \delta - \gamma + \beta - \alpha \geq \beta_2(X, \rho) - \beta_1(X, \rho) + \beta_0(X, \rho).$$

The inequality (16) implies that $\gamma - \beta + \alpha \geq B_1(X)$. Now the proposition follows from (17). \square

Corollary 3.20. *Assume that G is finite and the epimorphism $\pi_1(X) \rightarrow G$ is an isomorphism. Then*

$$\mu_2(\mathcal{C}_*(X)) \geq \delta(G) - b_1(X, \mathbb{F}) + b_2(X, \mathbb{F}).$$

Proof. It follows from Theorem 3.8 that $B_1(X) + b_0(X, \mathbb{F}) \geq \delta(G)$. \square

Remark 3.21. If G is a finite perfect group then $b_1(X, \mathbb{F}) = 0$, and $\delta(G) \geq 2$; thus we recover the Proposition 3.7.

Now we will give a lower bound for $\mu_2(X)$ in terms of a numerical invariant depending only on G and related to the invariant $D(G)$ (see Introduction). Up to the end of this Section we assume that G is finite and the epimorphism $\pi_1(X) \rightarrow G$ is an isomorphism. In this case the natural inclusion $\hat{\Lambda}[G] \hookrightarrow \mathbb{Z}[G]((T))$ is an isomorphism. We will make no difference between these two rings; observe also that

$$\mathcal{C}_*(X) = C_*(\tilde{X}) \otimes_{\mathbb{Z}} \hat{\Lambda}.$$

Definition 3.22. Let R be a commutative ring and

$$\mathcal{F}_* = \{0 \leftarrow R \leftarrow F_0 \leftarrow F_1 \leftarrow \dots\}$$

be a free $R[G]$ -resolution of the trivial $R[G]$ -module R ; put $m_i(\mathcal{F}_*) = m(F_i)$. The minimum of $m_i(\mathcal{F}_*)$ over all free resolutions of R will be denoted by $\mu_i(G, R)$.[†] If $R = \mathbb{Z}$ we abbreviate $\mu_i(G, R)$ to $\mu_i(G)$.

The following properties are easy to prove:

- 1) For any ring R we have $\mu_i(G, R) \leq \mu_i(G)$.
- 2) $\mu_1(G) = \delta(G)$.
- 3) $D(G) \geq \mu_1(G) + \mu_2(G)$.

We will now introduce a similar notion appearing in the context of \mathbb{Z} -graded complexes.

Definition 3.23. Let R be a commutative ring. A \mathbb{Z} -graded chain complex of free finitely generated $R[G]$ -modules

$$\mathcal{E}_* = \{\dots \leftarrow \mathcal{E}_{-n} \leftarrow \dots \leftarrow \mathcal{E}_0 \leftarrow \dots \mathcal{E}_n \leftarrow \dots\}$$

is called a \mathbb{Z} -graded resolution of the trivial $R[G]$ -module R if

- 1) $H_*(\mathcal{E}_*) = 0$ for every $i \neq 0$ and $H_0(\mathcal{E}_*) \approx R$.
- 2) $\mathcal{E}_{-n} = 0$ for every $n \geq 0$ sufficiently large.

The minimum of $m_i(\mathcal{E}_*)$ over all \mathbb{Z} -graded resolutions of R will be denoted by $\bar{\mu}_i(G, R)$. If $R = \mathbb{Z}$ we abbreviate $\bar{\mu}_i(G, R)$ to $\bar{\mu}_i(G)$.

We have obviously $\bar{\mu}_i(G, R) \leq \mu_i(G, R)$. The next proposition follows from the fundamental result of R. Swan [24].

Proposition 3.24. We have $\mu_i(G) = \bar{\mu}_i(G)$ for $i = 1, 2$.

Proof. Let \mathcal{E}_* be a \mathbb{Z} -graded resolution. Similarly to Proposition 3.5 we can assume that $\mathcal{E}_i = 0$ for $i \leq -2$; put

$$f_0 = m(\mathcal{E}_0) - m(\mathcal{E}_{-1}), \quad f_i = m(\mathcal{E}_i) \quad \text{for } i \geq 1.$$

Let $\rho : G \rightarrow \text{GL}(r, \mathbb{F})$ be any irreducible representation. Put $\mathcal{E}_*^\rho = \mathcal{E}_* \otimes \mathbb{F}^r$, and let Z_0^ρ be the vector space of cycles of degree 0. Then $\dim_\rho Z_0^\rho = r f_0$. By the strong Morse inequalities applied to the chain complex

$$0 \leftarrow Z_0^\rho \leftarrow \mathcal{E}_1^\rho \leftarrow \mathcal{E}_2^\rho \leftarrow \dots$$

we have

$$(18) \quad f_0 \geq \beta_0(\mathcal{E}_*, \rho);$$

$$(19) \quad f_1 - f_0 \geq \beta_1(\mathcal{E}_*, \rho) - \beta_0(\mathcal{E}_*, \rho);$$

$$(20) \quad f_2 - f_1 + f_0 \geq \beta_2(\mathcal{E}_*, \rho) - \beta_1(\mathcal{E}_*, \rho) + \beta_0(\mathcal{E}_*, \rho).$$

By the Swan's theory ([24], Th. 5.1, Corollary 6.1, and Lemma 5.2) there exists a free resolution \mathcal{F}_* of \mathbb{Z} over $\mathbb{Z}[G]$ such that $m_i(\mathcal{F}_*) = f_i$

[†] Our terminology here differs from that of the Swan's paper [24].

for $i = 0, 1, 2$. Therefore $\mu_i(G) \leq f_i$ for $i = 1, 2$. The proposition follows. \square

Remark 3.25. The proposition is valid for all $i \geq 1$, with some mild restrictions on G (see Theorem 5.1 of [24].)

A similar method proves the next proposition.

Proposition 3.26. $\bar{\mu}_i(G, \hat{\Lambda}) = \mu_i(G)$. \square

Now we can obtain the estimate for $\mu_2(\mathcal{C}_*(X))$.

Proposition 3.27. We have $\mu_2(\mathcal{C}_*(X)) \geq \bar{\mu}_2(G, \hat{\Lambda})$.

Proof. Let D_* be a free finitely generated chain complex chain equivalent to $\mathcal{C}_*(X)$. Then

$$H_i(D_*) = 0 \text{ for } i < 0, \text{ and } H_0(D_*) \approx \hat{\Lambda}, \text{ and } H_1(D_*) = 0.$$

Using the standard procedure of killing the homology groups of a chain complex, we embed D_* into a free chain complex $D'_* = D_* \oplus E_*$ such that D'_* is finitely generated in each dimension and

$$E_i = 0 \text{ for } i \leq 2, \text{ and } H_0(D'_*) = \hat{\Lambda} \text{ and } H_i(D'_*) = 0 \text{ for } i \neq 0.$$

Then $m_2(D_*) = m_2(D'_*) \geq \mu_2(G)$. The proposition follows. \square

The next Corollary is immediate.

Corollary 3.28. $\mu_2(\mathcal{C}_*(X)) \geq \mu_2(G)$. \square

4. INVARIANTS OF CHAIN COMPLEXES: $\mathbb{Z}/k\mathbb{Z}$ -GRADED CASE

Definition 4.1. Let R be a ring, and $k \in \mathbb{N}, k \geq 2$. A $\mathbb{Z}/k\mathbb{Z}$ -graded chain complex is a family of free based finitely generated R -modules A_i indexed by $i \in \mathbb{Z}/k\mathbb{Z}$ together with homomorphisms $\partial_i : A_i \rightarrow A_{i-1}$, satisfying $\partial_i \circ \partial_{i+1} = 0$.

Given $k \in \mathbb{N}$ and a free based finitely generated \mathbb{Z} -graded chain complex C_* , one constructs a $\mathbb{Z}/k\mathbb{Z}$ -graded chain complex C_*° as follows:

$$C_i^\circ = \bigoplus_{s \equiv i(k)} C_s.$$

In this section we will be working with the $\mathbb{Z}/k\mathbb{Z}$ -graded chain complex induced by $\mathcal{C}_*(X)$ (see the definition (11)). It will be denoted by $\mathcal{C}_*^\circ(X)$, where

$$(21) \quad \mathcal{C}_i^\circ(X) = \bigoplus_{s \equiv i(k)} \mathcal{C}_s(X).$$

Definition 4.2. Let C_* be a $\mathbb{Z}/k\mathbb{Z}$ -graded complex, and $i \in \mathbb{Z}/k\mathbb{Z}$. The minimal number $m(D_i)$ where D_* is a $\mathbb{Z}/k\mathbb{Z}$ -graded complex, chain equivalent to C_* , is denoted by $\mu_i(C_*)$.

4.1. Lower bounds from local coefficient homology. Similarly to section 3.2 we have the following estimates for the invariants μ_i of $\mathbb{Z}/k\mathbb{Z}$ -graded complexes. The proof of the next Proposition is similar to 3.3.

Proposition 4.3. Let $\rho : G \rightarrow \mathrm{GL}(r, \mathbb{F})$ a representation. Suppose that there is a group epimorphism $\pi_1(X) \rightarrow G$, then

$$\mu_i(\mathcal{C}_*^\circ(X)) \geq \sum_{s \in i(k)} \beta_s(X, \rho). \quad \square$$

4.2. Invariant μ_1 in the k -graded case. The previous theorem implies the following lower bounds for $\mu_i(\mathcal{C}_*^\circ(X))$ in terms of the invariants $d(G), \delta(G)$.

Theorem 4.4. Suppose that there exists a group epimorphism from $\pi_1(X)$ to a finite and non-trivial group G . Then

- 1) $\mu_1(\mathcal{C}_*^\circ(X)) \geq \max(\delta(G) - 1, 1)$.
- 2) If G is simple or solvable we have $\mu_0(\mathcal{C}_*^\circ(X)) + \mu_1(\mathcal{C}_*^\circ(X)) \geq d(G)$.

Proof. We need only to prove that $\mu_1(\mathcal{C}_*^\circ(X)) \geq 1$. To this end, observe that if $\mu_1(\mathcal{C}_*^\circ(X)) = 0$, then the homology of X in degree 1 with all local coefficients vanish, which imply $\delta(G) = 1$, then G is cyclic, $b_1(G) = 1$, which leads to a contradiction. \square

For the case of infinite groups we have the following result.

Theorem 4.5. Suppose that there exists a group epimorphism from $\pi_1(X)$ to an infinite group G . Then $\mu_1(\mathcal{C}_*^\circ(X)) \geq 1$.

Proof. Let D_* be a $\mathbb{Z}/k\mathbb{Z}$ -graded chain complex equivalent to $\mathcal{C}_*^\circ(X)$. The module $\hat{\Lambda} = H_0(\mathcal{C}_*(X))$ is a submodule of $H_0(\mathcal{C}_*^\circ(X))$. The condition $D_1 = 0$ would imply that $H_0(\mathcal{C}_*^\circ(X))$ and $\hat{\Lambda}$ are submodules of a free \mathcal{L} -module $\mathcal{C}_0^\circ(X)$, and this is impossible when G is infinite by Lemma 3.12. \square

This theorem holds for $\hat{\Lambda} \otimes_{\mathbb{Z}} \mathbb{Q}$ by the same argument. This estimate can be improved in the case when $k - 2 \geq \dim X$. Note that in this case the sum in the right hand side of (21) contains only one term for every i .

Theorem 4.6. Assume that $\dim X \leq k - 2$, and there exists a group epimorphism from $\pi_1(X)$ to a finite group G . Then

$$\mu_1(\mathcal{C}_*^\circ(X)) \geq \delta(G).$$

Proof. Similarly to Section 3.3 it suffices to prove that

$$\mu_1(\mathcal{C}_*^\circ(X)) \geq \beta_1(X, \rho) + 1$$

for any representation ρ such that $H_0(X, \rho) = 0$. Let

$$D_* = \{\dots \leftarrow D_{-2} \xleftarrow{\partial_{-1}} D_{-1} \xleftarrow{\partial_0} D_0 \leftarrow \dots\}$$

be a $\mathbb{Z}/k\mathbb{Z}$ -graded complex, chain equivalent to $K_* = \mathcal{C}_*^\circ(X)$. The chain complex D_* does not necessarily vanish in any degree, and the argument which we used in the proof of the Proposition 3.5 can not be applied immediately.

Let $D_* \xrightarrow{\phi} K_* \xrightarrow{\psi} D_*$ be the mutually inverse chain equivalences. Since $k \geq \dim X + 2$, the chain complex K_* vanishes in degree -1 , hence the map $\psi \circ \phi : D_{-1} \rightarrow D_{-1}$ is equal to 0. The existence of chain homotopy from $\psi \circ \phi$ to Id implies that the submodule $\text{Ker } \partial_{-1} = \text{Im } \partial_0$ is a direct summand of D_{-1} , hence a projective $\hat{\Lambda}[G]$ -module. Let us denote it by L . The $\mathbb{Z}/k\mathbb{Z}$ -graded chain complex D_* contains a (\mathbb{Z} -graded) subcomplex

$$D'_* = \{0 \leftarrow L \xleftarrow{\partial_0} D_0 \leftarrow D_1 \leftarrow D_2 \leftarrow 0\},$$

with $H_0(D'_*) \approx \hat{\Lambda}$, $H_{-1}(D'_*) = 0$, $H_1(D'_*) \approx H_1(D_*)$. Denote by $S \subset \mathbb{N}$ the multiplicative subset of all numbers t , such that $\gcd(t, |G|) = 1$. The module $S^{-1}L$ is free by a fundamental result of R. Swan (see [21], §5). Thus the chain complex $D''_* = S^{-1}D'_*$ is free over $S^{-1}\hat{\Lambda}[G]$; put

$$\alpha = m(D''_{-1}), \beta = m(D''_0), \gamma = m(D''_1).$$

Let p be a prime divisor of $|G|$, and $\rho : G \rightarrow \text{GL}(r, \mathbb{F}_p)$ be a representation. The homology of the complex $K_* \otimes_{\rho} \mathbb{F}_p^r$ is isomorphic to that of $D''_* \otimes_{\rho} \mathbb{F}_p^r$ in degrees $-1, 0, 1$. Therefore the argument proving Proposition 3.5 applies here as well, and the proof of the Theorem is complete. \square

5. ESTIMATES FOR THE NUMBER OF CLOSED ORBITS

We proceed to the estimates of the number of periodic orbits of a Hamiltonian isotopy induced by a non-degenerate 1-periodic Hamiltonian H on a closed connected symplectic manifold M . We denote by $\tilde{M} \rightarrow M$ a regular covering with a structure group G . Put

$$(22) \quad \mathcal{C}_*(M) = C_*(\tilde{M}) \otimes_{\mathbb{Z}[G]} \Lambda_{(M, \omega)}^{\mathbb{Z}[G]}.$$

Denote by N the minimal Chern number of M .

Definition 5.1. If $N = 0$, put

$$(23) \quad \mu_i(\widetilde{M}) = \mu_i(\mathcal{C}_*(M)).$$

(see Definition 3.1; here $i \in \mathbb{N}$).

If $N > 0$ consider the $\mathbb{Z}/2N\mathbb{Z}$ -graded chain complex $C_*^\circ(\widetilde{M})$ (see (21)) and put

$$(24) \quad \mu_i(\widetilde{M}) = \mu_i(\mathcal{C}_*^\circ(M)).$$

(see Definition 4.2; here $i \in \mathbb{Z}/2N\mathbb{Z}$).

The numbers $\mu_i(\widetilde{M})$ are obviously homotopy invariants of M and the chosen covering $\widetilde{M} \rightarrow M$.

5.1. The spherical Calabi-Yau case. We consider here symplectic manifolds M with $c_1(M)(A) = 0$ for every $A \in \pi_2(M)$. In this case every contractible periodic orbit γ has a well-defined index $i(\gamma) \in \mathbb{Z}$. The Floer chain complex \widetilde{CF}_* is a \mathbb{Z} -graded free based finitely generated chain complex over the ring $\Lambda_{(M,\omega)}^{\mathbb{Z}[G]}$, generated in degree k by contractible periodic orbits of the Hamiltonian vector field of index k , and we have

$$\widetilde{CF}_* \sim \mathcal{C}_{*+n}(M), \text{ where } \dim M = 2n.$$

Denote by p_k the number of contractible periodic orbits of period k . The results of the previous sections imply the following lower bound:

$$p_{i-n} \geq \mu_i(\widetilde{M}).$$

Applying the results of Section 3 we obtain the following lower bounds for p_i :

Proposition 5.2. *For any field \mathbb{F} and any representation $\rho : G \rightarrow \text{GL}(r, \mathbb{F})$ we have*

$$p_{i-n} \geq \frac{1}{r} b_i(M, \rho).$$

Theorem 3.10 and Corollary 3.11 imply some stronger lower bounds for $i = 1$.

Theorem 5.3. 1) *If $\pi_1(M)$ is non-trivial, then $p_{1-n} \geq 1$.*

2) *If $\pi_1(M)$ has an epimorphism onto a finite group G , then*

- (a) $p_{1-n} \geq \delta(G)$,
- (b) $p_{1-n} \geq d(G)$ if G is solvable or simple,
- (c) $p_{1-n} \geq 2$ if G is not cyclic.

Let us proceed to the lower bounds for p_{2-n} . Applying Corollaries 3.20 and 3.28 we obtain the following result.

Theorem 5.4. *Assume that $\pi_1(M)$ is finite and the homomorphism $\pi_1(M) \rightarrow G$ is an isomorphism. Then*

- 1) $p_{2-n} \geq \delta(\pi_1(M)) - b_1(M, \mathbb{F}) + b_2(M, \mathbb{F})$ for any field \mathbb{F} .
- 2) If G is perfect, then $p_{2-n} \geq b_2(M, \mathbb{F}) + 2$.
- 3) $p_{2-n} \geq \mu_2(\pi_1(M))$.

Remark 5.5. A recent result of Joel Fine and Dmitry Panov [7] asserts that for every finitely presented group G there exists a symplectic manifold M of dimension 6 with the fundamental group G and $c_1(M) = 0$.

5.2. The weakly monotone case. Let us denote by p the total number of the periodic orbits of the Hamiltonian vector field. We have a $\mathbb{Z}/2N\mathbb{Z}$ -graded chain complex \widetilde{CF}_* , generated by periodic orbits, such that

$$\widetilde{CF}_* \sim \mathcal{C}_{*+n}^\circ(M), \text{ where } 2n = \dim M,$$

(see Theorem 2.8). Therefore

$$p_{i-n} \geq \mu_i(\widetilde{M}) \text{ for } i \in \mathbb{Z}/2N\mathbb{Z}.$$

Applying the results of the section 4 we obtain the following.

Theorem 5.6. *For every representation $\rho : G \rightarrow \mathrm{GL}(r, \mathbb{F})$ we have*

$$p_{i-n} \geq \frac{1}{r} \left(\sum_{s \equiv i(2N)} b_s(M, \rho) \right) \text{ for } i \in \mathbb{Z}/2N\mathbb{Z}.$$

As for the number p_{1-n} we have the following.

- Theorem 5.7.**
- 1) If $\pi_1(M)$ is non-trivial, then $p_{1-n} \geq 1$.
 - 2) If $\pi_1(M)$ has an epimorphism onto a finite group G , then
 - (a) $p_{1-n} \geq \max(1, \delta(G) - 1)$, and $p \geq \delta(G)$.
 - (b) if G is simple or solvable, then $p \geq d(G)$,
 - (c) if G is not cyclic, then $p \geq 2$.

For the manifolds where the minimal Chern number N is strictly greater than $n = \dim M/2$ we have the following improvement of Theorem 5.7 (the proof follows from Theorem 4.6):

Theorem 5.8. *Let $N \geq n + 1$. Assume that $\pi_1(M)$ has an epimorphism onto a finite group G . Then*

- 1) $p_{1-n} \geq \delta(G)$.
- 2) If G is simple or solvable, then $p_{1-n} \geq d(G)$.
- 3) If G is not cyclic, then $p_{1-n} \geq 2$.

5.3. The general case. Let M^{2n} be an arbitrary closed connected symplectic manifold. Theorem 2.9 together with Proposition 3.14, Theorem 4.5 implies the following result. As we noted, Theorem 4.5 holds for $\hat{\Lambda} \otimes_{\mathbb{Z}} \mathbb{Q}$.

Theorem 5.9. *Assume that $\pi_1(M)$ is infinite. Then $p_{1-n} \geq 1$.*

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